University of Ottawa
MAT 1332D Midterm Exam
March 25, 2009. Duration: 80 minutes. Instructor: Jing Li

Family Name:______________________

First Name:_______________________

Do **not** write you student ID number on this front page. Please write your student ID number in the space provided on the second page.

Take your time to read the entire paper before you begin to write, and read each question carefully. Remember that certain questions are worth more points than others. Make a note of the questions that you feel confident you can do, and then do those first: <u>you do not</u> have to proceed through the paper in the order given.

- You have 80 minutes to complete this exam.
- This is a closed book exam, and no notes of any kind are allowed. The use of cell phones, pagers or any text storage or communication device **is not permitted.**
- Only the Faculty approved TI-30 calculator is allowed.
- The correct answer requires justification written legibly and logically: you must convince me that you know why your solution is correct. Answer these questions in the space provided. Use the backs of pages if necessary.
- Where it is possible to check your work, do so.
- Good Luck!

Student number:____________________, Total marks: ___________ out of 32

Problem	1	2	3	4	5
Marks					

Question 1. [12 points] Consider the following matrix:

$$A = \begin{bmatrix} 4 & 0 & 2 \\ 0 & -2 & 0 \\ 8 & 0 & 3 \end{bmatrix}.$$

1. [2 points] Calculate the determinant and explain why the matrix is invertible. (One short sentence is enough.)
2. [4 points] Find A^{-1}.
3. [1 point] Solve the equation $Ax = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}$.
4. [3 points] Show that $\lambda_1 = -2$ is an eigenvalue of A and find the other two eigenvalues.
5. [2 points] Find the eigenvectors corresponding to $\lambda_1 = -2$.

Solution:

1. The determinant of Matrix A is given by

$$\det(A) = 4 \cdot (-2) \cdot 3 - 2 \cdot (-2) \cdot 8 = -24 + 32 = 8.$$

The matrix is invertible is because the determinate of Matrix A is not zero.

2. A^{-1}, the inverse of Matrix A, can be calculated as follows,

$$\left[\begin{array}{ccc|ccc} 4 & 0 & 2 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & 1 & 0 \\ 8 & 0 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-2R_1 + R_3} \left[\begin{array}{ccc|ccc} 4 & 0 & 2 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -2 & 0 & 1 \end{array}\right] \xrightarrow{2R_3 + R_1}$$

$$\left[\begin{array}{ccc|ccc} 4 & 0 & 0 & -3 & 0 & 2 \\ 0 & -2 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -2 & 0 & 1 \end{array}\right] \xrightarrow{\substack{\frac{1}{4}R_1 \\ -\frac{1}{2}R_2 \\ -R_3}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{3}{4} & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 2 & 0 & -1 \end{array}\right]$$

So

$$A^{-1} = \begin{bmatrix} -\frac{3}{4} & 0 & \frac{1}{2} \\ 0 & -\frac{1}{2} & 0 \\ 2 & 0 & -1 \end{bmatrix}.$$

3. The solution of the equation $Ax = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}$ is given by

$$x = A^{-1} \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -3+1 \\ -1 \\ 8-2 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 6 \end{bmatrix}.$$

4. The eigenvalue of Matrix A can be calculated as follows:

   First we from the matrix

$$A - \lambda I = \begin{bmatrix} 4 & 0 & 2 \\ 0 & -2 & 0 \\ 8 & 0 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} 4-\lambda & 0 & 2 \\ 0 & -2-\lambda & 0 \\ 8 & 0 & 3-\lambda \end{bmatrix}.$$

   Then we calculate its determinant as

$$\det(A) = (4-\lambda)\cdot(-2-\lambda)\cdot(3-\lambda) - 2\cdot(-2-\lambda)\cdot 8 = (2+\lambda)\big[16 - (4-\lambda)(3-\lambda)\big]$$

   Hence, $2 + \lambda = 0$ gives the eigenvalue $\lambda_1 = -2$. Solving

$$16 - (4-\lambda)(3-\lambda) = 0, \quad i.e., \quad \lambda^2 - 7\lambda - 4 = 0,$$

   gives the other two eigenvalues

$$\lambda_2 = \frac{7+\sqrt{65}}{2}, \quad \lambda_3 = \frac{7-\sqrt{65}}{2}.$$

5. For $\lambda_1 = -2$, we solve the system

$$\begin{bmatrix} 6 & 0 & 2 \\ 0 & 0 & 0 \\ 8 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$$

   To write it as only the coefficient matrix and suppress the right hand column of zeros,

$$\begin{bmatrix} 6 & 0 & 2 \\ 0 & 0 & 0 \\ 8 & 0 & 5 \end{bmatrix} \xrightarrow{-\frac{4}{3}R_1 + R_3} \begin{bmatrix} 6 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{7}{3} \end{bmatrix}$$

   The second column gives $0 \cdot x_2 = 0$, which means $x_2 = t$ is free. The third row reads $\frac{7}{5}x_3 = 0$, which means $x_3 = 0$. Sub $x_3 = 0$ into $6x_1 + 2x_3 = 0$ given by the first row, we obtain $x_1 = 0$. Hence we get the eigenvector

$$v_1 = t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{with} \quad t \neq 0.$$

Question 2. [6 points] Consider the system of linear equations

$$\begin{aligned} x + ay &= 2 \\ bx + 3y &= 4 \end{aligned}$$

where a and b are parameters.

1. [2 points] Determine the condition(s) on a and b to get a unique solution.
2. [2 points] Determine the condition(s) on a and b to get infinitely many solutions.
3. [2 points] Determine the condition(s) on a and b such that the system has no solutions.

Solution: To write the system in the format of the augmented matrix gives

$$\left[\begin{array}{cc|c} 1 & a & 2 \\ b & 3 & 4 \end{array}\right] \xrightarrow{-bR_1 + R_2} \left[\begin{array}{cc|c} 1 & a & 2 \\ 0 & 3-ab & 4-2b \end{array}\right].$$

1. If $ab - 3 \neq 0$, we can continue to do

   $$\left[\begin{array}{cc|c} 1 & a & 2 \\ 0 & 3-ab & 4-2b \end{array}\right] \xrightarrow{\frac{1}{3-ab}R_2} \left[\begin{array}{cc|c} 1 & a & 2 \\ 0 & 1 & \frac{4-2b}{3-ab} \end{array}\right] \xrightarrow{-aR_2 + R_1} \left[\begin{array}{cc|c} 1 & 0 & 2 - a \cdot \frac{4-2b}{3-ab} \\ 0 & 1 & \frac{4-2b}{3-ab} \end{array}\right].$$

   Hence, there is a unique solution given by

   $$x = \frac{6-4a}{3-ab}, \quad y = \frac{4-2b}{3-ab}.$$

   The condition on a and b to get a unique solution of the system is $ab - 3 \neq 0$.

2. If $3 - ab = 4 - 2b = 0$ i.e., $a = \frac{3}{2}, b = 2$, we get the system

   $$\left[\begin{array}{cc|c} 1 & a & 2 \\ 0 & 0 & 0 \end{array}\right]$$

   There are infinitely many solutions of the form

   $$\{(2 - at, t) : t \in \mathbb{R}\}.$$

   So the conditions on a and b to get infinitely many solutions are $a = \frac{3}{2}, b = 2$.

3. If $3 - ab = 0$ but $4 - 2b \neq 0$, there is no solution because the last row of the augmented matrix yields $0 = 4 - 2b \neq= 0$, which is impossible.

   So the conditions on a and b on a and b such that the system has no solutions are $3 - ab = 0$ and $b \neq 2$.

Question 3. [5 points] Kitty the bird lives in Hawaii, where she travels between the islands of Maui (M) and Big Island (B). People tell you that Kitty's movement between M and B can be modeled as a Markov chain with the transition matrix

$$P = \begin{bmatrix} 0.4 & 0.3 \\ 0.6 & 0.7 \end{bmatrix},$$

where 0.4 is the probability that Kitty will stay in M next week if she is there this week.

1. [2 points] Assume that Kitty is on M this week. What is the probability that she is on M in two weeks?

2. [3 points] What is the percentage of time that Kitty spends on M in the long run?

Solution: Let $x^{(n)}$ and $y^{(n)}$ be the probability of Kitty is on M and on B in n weeks, respectively.

1. The assumption that Kitty is on M this week gives the initial condition of Markov Chain as

$$\begin{bmatrix} x^{(0)} \\ y^{(0)} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

So the probabilities that Kitty is on M and on B in the second week, respectively, can be calculated as follows,

$$\begin{bmatrix} x^{(1)} \\ y^{(1)} \end{bmatrix} = \begin{bmatrix} 0.4 & 0.3 \\ 0.6 & 0.7 \end{bmatrix} \begin{bmatrix} x^{(0)} \\ y^{(0)} \end{bmatrix} = \begin{bmatrix} 0.4 & 0.3 \\ 0.6 & 0.7 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix}.$$

To continue further, we can obtain the probability of Kitty in on M and on B in two weeks as follows,

$$\begin{bmatrix} x^{(2)} \\ y^{(2)} \end{bmatrix} = \begin{bmatrix} 0.4 & 0.3 \\ 0.6 & 0.7 \end{bmatrix} \begin{bmatrix} x^{(1)} \\ y^{(1)} \end{bmatrix} = \begin{bmatrix} 0.4 & 0.3 \\ 0.6 & 0.7 \end{bmatrix} \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix} = \begin{bmatrix} 0.4 \cdot 0.4 + 0.3 \cdot 0.6 \\ 0.6 \cdot 0.4 + 0.7 \cdot 0.6 \end{bmatrix} = \begin{bmatrix} 0.34 \\ 0.67 \end{bmatrix}.$$

Hence assume that Kitty is on M this week, the probability that she is on M in two weeks is 0.34.

2. The steady state of this Markov chain will answer this question. To solve

$$\begin{bmatrix} x \\ y \end{bmatrix} = P \begin{bmatrix} x \\ y \end{bmatrix}$$

i.e.,

$$\begin{bmatrix} 0.4 - 1 & 0.3 \\ 0.6 & 0.7 - 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0.$$

Then we know that

$$-0.6x + 0.3y = 0.$$

Together with the condition that $x + y = 1$, we have $x = 1/3$ and $y = 2/3$. Hence the percentage of time that Kitty spends on M in the long run is 33.3%.

Question 4. [4 points] Consider the following function of two variables:

$$f(x,y) = 1 - \frac{2y}{x} + 3x - 4x^2y^3 + e^{5y}.$$

1. [2 points] Find the partial derivatives of f with respect to x and y.
2. [2 points] Find the linear approximation at the point $(x,y) = (1,0)$.

Solution:

1. The partial derivative of f with respect to x is given by

$$f_x = \frac{\partial f}{\partial x} = \frac{2y}{x^2} + 3 - 8xy^3.$$

The partial derivative of f with respect to y is given by

$$f_y = \frac{\partial f}{\partial y} = -\frac{2}{x} - 12x^2y^2 + 5e^{5y}.$$

2. For $(x,y) = (1,0)$, we have

$$f(1,0) = 1 - 0 + 3 \cdot 1 - 0 + e^0 = 5,$$

$$f_x(1,0) = \frac{\partial f}{\partial x}(1,0) = 0 + 3 - 0 = 3,$$

$$f_y(1,0) = \frac{\partial f}{\partial y}(1,0) = -2 - 0 + 5e^0 = 3.$$

The linear approximation of $f(x,y) = 1 - \frac{2y}{x} + 3x - 4x^2y^3 + e^{5y}$ at the point$(x,y) = (1,0)$ is given by

$$\begin{aligned} f(x,y) &\approx f(1,0) + \frac{\partial f}{\partial x}(1,0)(x-1) + \frac{\partial f}{\partial y}(1,0)(y-0) \\ &= 5 + 3(x-1) + 3y \end{aligned}$$

i.e.,

$$f(x,y) \approx 3x + 3y + 2.$$

Question 5. [5 points] Consider the equation $x^3 - 5x^2 + 9x - 5 = 0$.

1. [1 point] Show that $x_1 = 1$ is a solution of the equation.
2. [2 points] Use long division to show that the other two roots are $x_2 = 2-i$ and $x_3 = 2+i$.
3. [1 point] Calculate $x_2 x_3$.
4. [1 point] You can choose to do **either (a) or (b)**:

   NOTE: You can earn this point for Part 4 if you do one of them correctly. You **will not** get bonus points if you do both correctly. So please choose the one you are more confident with.

   (a) Calculate $\frac{x_2}{x_3}$.

   (b) Express x_3 in the form $x_3 = re^{i\theta}$.

Solution:

1. Sub $x_1 = 1$ we have
$$1^3 - 5 \cdot 1^2 + 9 \cdot 1 - 5 = 0.$$
This implies that $x_1 = 1$ is a solution of the equation.

2. From Part 1, we know that $x - 1$ will be one of the factors of $x^3 - 5x^2 + 9x - 5$. By long division, we find that
$$x^3 - 5x^2 + 9x - 5 = (x-1)(x^2 - 4x + 5).$$
So solving $x^2 - 4x + 5 = 0$ gives the other two roots of the equation $x^3 - 5x^2 + 9x - 5 = 0$ as follows,
$$x_2 = 2 - i, \quad x_3 = 2 + i.$$

3. $x_2 x_3 = (2-i)(2+i) = 4 + 2i - 2i + i^2 = 3$.

4. (a)
$$\frac{x_2}{x_3} = \frac{2-i}{2+i} = \frac{(2-i)(2-i)}{(2+i)(2-i)} = \frac{4 - 2i - 2i - i^2}{4 - i^2} = \frac{3-4i}{5}.$$

   (b) Note that $r = |x_3| = \sqrt{2^2 + 1^2} = \sqrt{5}$, and $\theta = \arctan\frac{1}{2}$ (see Figure 1 for illustration), we have
$$x_3 = \sqrt{5} e^{i \arctan\frac{1}{2}}.$$

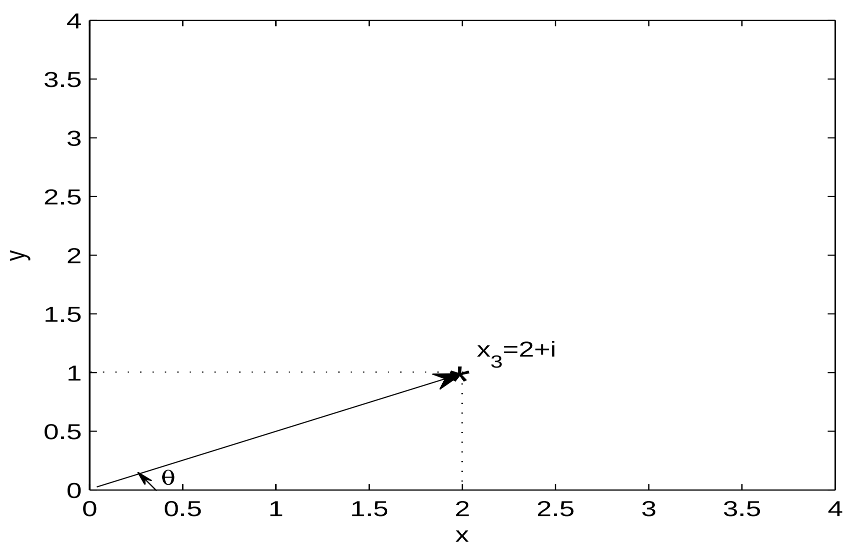

Figure 1: $x_3 = 2 + i$.